Stability analysis of phonon transport equations derived via the Chapman-Enskog method and transformation of variables

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Under the assumption of Callaway's model of the Boltzmann-Peierls equation, the Chapman-Enskog method for a phonon gas forms the basis to derive various hydrodynamic equations for the energy density and the drift velocity of interest when normal processes dominate over resistive ones. The first three levels of the expansion (i.e., the zeroth-, first-, and second-order approximations) are satisfactory in that they are entropy consistent and ensure linear stability of the rest state. However, the entropy density contains a weakly nonlocal term, the entropy production is a degenerate function of variables, and the next order in the Chapman-Enskog expansion gives the equations with linearly unstable rest solutions. In the context of Burnett and super-Burnett equations, a similar type of problem was recognized by several authors who proposed different ways to deal with it. Here we report on yet another possible device for obtaining more satisfactory equations. Namely, inspired by the fact that there exists no unique way to truncate the Chapman-Enskog expansion, we combine the Chapman-Enskog procedure with the method of variable transformation and subsequently find a class of ε-dependent transformations through which it is possible to derive the second-order equations possessing a local entropy density and nondegenerate expression for the entropy production. Regardless of this result, we also show that although the method cannot be used to construct linearly stable third-order equations, it can be used to make the originally stable first-order equations asymptotically stable.

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I. INTRODUCTION

The Chapman-Enskog method as applied to a phonon gas provides a solution to the Boltzmann-Peierls equation [1,2] in which the distribution function is expressed in the form of a power series in the small parameter ε. Several temperature ranges may be defined according to the relation between the effective relaxation times for normal and resistive processes [3]. In the case when normal processes dominate over resistive ones, from the solution of the Boltzmann-Peierls equation using this method it is possible to derive various nonlinear systems of hydrodynamic equations for the energy density and the drift velocity [4,5]. These systems are nonlinear because no limitations were introduced on the magnitudes of the individual components of the drift velocity or the heat flux. For one-dimensional flow problems, restricting attention to Callaway's model of the collision operator [6] and assuming nondispersion and isotropy in the frequency spectrum, it was explicitly demonstrated that the first three levels of the expansion (i.e., the zeroth-, first-, and second-order approximations) yield the equations of hydrodynamics which are linearly stable at all wavelengths. The next order in the Chapman-Enskog expansion leads to equations which are unstable to some perturbations. More explicitly, the linearized equations of motion that describe the propagation of small disturbances in the flow have unstable plane-wave solutions in the short-wavelength limit of the dispersion relations [5].

Similar problems are encountered when considering ideal monoatomic gases described by the Boltzmann equation. There, the results of the computations made for Maxwellian molecules and rigid spheres show that the usual Chapman-Enskog procedure does not work already at the level of Burnett equations (the next step after the Navier-Stokes equations) [7,8]. Precisely speaking, the second-order Burnett approximation implies that sufficiently short acoustic waves are increasing with time instead of decaying. As explained by Bobylev [9,10], the main reason for the existence of this short-wavelength instability is a violation of the condition ensuring that the matrix of the coefficients for third spatial derivatives has only real eigenvalues. This condition is not fulfilled in most typical cases, the prominent exception being the case of Burnett equations derived from the standard Bhatnagar-Gross-Krook model [11]. Due to the difficulties associated with the Burnett equations as described above, several modified approaches have been suggested by many investigators in the last two decades [12–18]. These approaches are mostly based on a combination of the Chapman-Enskog method with moment methods and on using some higher order in ε terms for regularization of the Burnett equations. However, the recent method of Bobylev [9,10] does not use any information beyond the Burnett level of approximation and is based on the following concept of transformation of variables. Namely, this concept is enlarged to encompass the situation where the transformation operators themselves depend explicitly on the small parameter ε. The power series expansions of these operators offer four

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advantages: (i) they yield the transformation of variables in a more explicit form; (ii) in a function of the original variables, substitution of the new variables consists simply of an iterative procedure involving only explicit chains of commutators; (iii) the inverse transformation can be built exactly the same way; (iv) there exist the transformation operators such that the Burnett equations formulated in terms of the new variables are linearly stable at all wavelengths.

Note that, besides Bobylev's two papers mentioned above, the idea to see how the Chapman-Enskog procedure may change with a change in variables was also studied in the two papers by Colangeli *et al.* [19,20]. These authors discussed a simple kinetic model, namely, the linearized Grad's 13 moment equations. Their approach is based on a dynamic invariance principle which gives exact constitutive relations for the stress tensor and heat flux, and a transformation which makes the exact equations of hydrodynamics stable and consistent with the second law of thermodynamics.

Proceeding along the lines suggested by Bobylev [9,10], the aim of this paper is to study the issue of stability of the first-, second-, and third-order phonon hydrodynamic equations derived via the Chapman-Enskog method combined with ε-dependent transformation of variables. As a starting point, we consider the infinite system of moment equations consistent with, and equivalent to, Callaway's model of the Boltzmann-Peierls equation. For one-dimensional flow problems, this system is linearized near the equilibrium state (i.e., the state with zero background drift velocity), assuming that the effective relaxation time for normal processes is much smaller than the effective relaxation time for resistive processes. The Chapman-Enskog expansion in terms of the small parameter ε on such moment equations yields the systems of linear partial differential equations for describing the dynamics of small departures of the phonon gas from its equilibrium state. An important observation is that these systems are not unique because the result of truncation at each order in ε depends on the definition of the perturbation variables. In our previous paper [5], this definition was standard, which means that it was independent of the small parameter ε. Consequently, truncation at the levels $O(\varepsilon^1)$ and $O(\varepsilon^2)$ implied stability of the equilibrium state, but not the existence of a nondegenerate local H theorem. Here we show that by using the Chapman-Enskog method combined with a variable transformation approach, it is possible to derive the equations which possess the property of yielding such an H theorem. Regardless of this result, we also study the problem of boundary conditions in order to show that the above approach can be used to make the originally stable first-order equations asymptotically stable. Matters are quite comfortable up to this point but when we go to higher orders, we first encounter the equations associated with the super-Burnett level of approximation. Since these equations are unstable and the equations after transformation are also unstable, we must conclude that the method does not avoid the failures of stability.

The layout of this paper is as follows. Under the assumption of the one-dimensional rotationally symmetric geometry, Sec. II is devoted to a treatment of the more relevant aspects of Callaway's model. Section III introduces the moment equations corresponding to Callaway's model; these equations are subsequently linearized around the equilibrium state. Section IV describes the Chapman-Enskog procedure. Section V considers the idea of change in variables. Section VI states the initial-boundary value problem for Eq. (5.1) with $p=1$ and subsequently discusses the notion of asymptotic stability for this equation. The Appendix defines the spaces used in Sec. VI.

Throughout our work, we employ units which are defined by setting $\hbar=k_B=1$. For the sake of simplicity, we use the Boltzmann-Peierls equation with Callaway's collisional term. The discussion is mostly restricted to the case when the effective relaxation time for normal processes is a constant quantity and the effective relaxation time for resistive processes equals infinity [21]. No distinction is made between longitudinal and transverse phonons. The dispersion relation has the form $\omega_D=c|\mathbf{k}|$, where c is the constant Debye speed. We let the components $(k_i)=(k_1,\ldots,k_n)$ of the wave vector $\mathbf{k}$ range from $-\infty$ to ∞. As to the value of n in $(k_1,\ldots,k_n)$, we consider exclusively the spacetime whose spatial dimension is either two or three; thus it is assumed that $n=2$ or $n=3$.

II. KINETIC MODEL

The basic equation of phonon kinetic theory is the Boltzmann-Peierls equation [1]. Assuming the one-dimensional rotationally symmetric geometry, this equation governs the time evolution in phase space of the distribution function $f(t,x,|\mathbf{k}|,k_x)$ describing the number of phonons at position x having wave number in the x direction $k_x:=k_1$ and is given as

$$\partial_t f + cg\partial_x f = J_R(f) + J_N(f),$$

where $g\in[-1,1]$ is the x component of $\mathbf{k}/|\mathbf{k}|$,

$$g := k_x/|\mathbf{k}| = \begin{cases} k_1/\sqrt{(k_1)^2+(k_2)^2} & \text{if } n=2, \\ k_1/\sqrt{(k_1)^2+(k_2)^2+(k_3)^2} & \text{if } n=3, \end{cases}$$

and $J_R(f)$ and $J_N(f)$ stand for the collision terms due to resistive and normal processes, respectively. For the sake of simplicity, we consider only the relaxation-time model of the collision terms. As discussed by Callaway [6], this involves the use of $J_R(f)$ and $J_N(f)$ of the form

$$J_R(f) = \frac{1}{\tau_R}(F_0 - f), \quad J_N(f) = \frac{1}{\tau_N}(F - f)$$

as the starting point. Here (τ_R,τ_N) are the effective relaxation times for resistive and normal processes, which we assume are the constant quantities, and (F_0,F) are the equilibrium and displaced Planck distributions defined by

$$F_0 := \frac{y}{\exp(ck/T_0)-1}, \quad F := \frac{y}{\exp[(ck/T)(1-vg)]-1},$$

where

$$y := n(2\pi)^{-n}, \quad k := |\mathbf{k}|, \quad -1 < v < 1.$$

The functions $T_0=T_0(t,x)$ and $T=T(t,x)$ represent two different temperature fields and the function $v=v(t,x)$ represents

the dimensionless drift velocity of a phonon gas. We fix these functions so as to reproduce the actual energy density $e=e(t,x)$ and the actual heat flux $q=q(t,x)$:

$$c\int kF_0 d^n\mathbf{k} = c\int kFd^n\mathbf{k} = e := c\int kfd^n\mathbf{k}, \quad (2.1a)$$

$$c^2\int k_x Fd^n\mathbf{k} = q := c^2\int k_x fd^n\mathbf{k}. \quad (2.1b)$$

Conditions (2.1) make it possible to relate T_0 to e and (T,v) to (e,q).

Now, observing that $f(t,x,|\mathbf{k}|,k_x)$ can also be written as $f(t,x,k,kg)$, it will be convenient to introduce the distribution function which depends on (t,x,g) and does not depend on k:

$$\phi := c\pi\int_0^\infty k^n f(t,x,k,kg)dk. \quad (2.2)$$

We call ϕ the reduced distribution function. With the aid of Eq. (2.2), using the notation

$$\psi_0 := c\pi\int_0^\infty k^n F_0 dk, \quad \psi := c\pi\int_0^\infty k^n F dk,$$

the relaxation-time model yields the following equation for ϕ:

$$\partial_t\phi + cg\partial_x\phi = \frac{1}{\tau_R}(\psi_0-\phi) + \frac{1}{\tau_N}(\psi-\phi). \quad (2.3)$$

Conditions (2.1) transform to

$$2(n-1)\int \psi_0 dg = 2(n-1)\int \psi dg = e := 2(n-1)\int \phi dg, \quad (2.4a)$$

$$2(n-1)c\int g\psi dg = q := 2(n-1)c\int g\phi dg, \quad (2.4b)$$

where

$$\int \mathcal{F}(g)dg := \frac{1}{2\pi^{3-n}}\int_0^{(4-n)\pi} \mathcal{F}(\cos\theta)\sin^{n-2}\theta d\theta,$$

with g given by $g=\cos\theta$.

As in Ref. [5], we take Eq. (2.3) with the formal small parameter ε inserted:

$$\partial_t\phi + cg\partial_x\phi = \frac{\varepsilon}{\tau_R}(\psi_0-\phi) + \frac{1}{\varepsilon\tau_N}(\psi-\phi). \quad (2.5)$$

This form of the equation for ϕ may serve as a basis for developing the Chapman-Enskog procedure of interest when normal processes dominate over resistive ones. Finally, we use the constant relaxation time τ_N and the constant Debye speed c to nondimensionalize the time and space coordinates according to

$$t = \tau_N t', \quad x = c\tau_N x'.$$

Thus, Eq. (2.5) is equivalent to

$$\partial_{t'}\phi + g\partial_{x'}\phi = \varepsilon\eta(\psi_0-\phi) + \frac{1}{\varepsilon}(\psi-\phi), \quad (2.6)$$

where $\eta := \tau_N/\tau_R$. The primes on t and x will be omitted in the sequel.

III. MOMENT EQUATIONS

A. General form

The equations for (e,v) emerging from Eqs. (2.4) and (2.6) are of the form

$$\partial_t e + \frac{(n+1)v}{n+u}\partial_x e + \frac{(n+1)(n-u)e}{(n+u)^2}\partial_x v = 0, \quad (3.1a)$$

$$\partial_t v + \frac{n(1-u)^2}{(n+1)(n-u)e}\partial_x e + \frac{2(n-3)v+(3n-1)uv}{n^2-u^2}\partial_x v + \frac{(n+u)^2}{(n+1)(n-u)e}(e\partial_x\lambda_2 + \lambda_2\partial_x e) = -\frac{\varepsilon\eta(n+u)v}{n-u}, \quad (3.1b)$$

where $u := v^2$ and λ_2 represents the dimensionless second-order moment of $\phi-\psi$:

$$\lambda_2 := \frac{2(n-1)}{e}\int\left(g^2-\frac{1}{n}\right)(\phi-\psi)dg. \quad (3.2)$$

We define the dimensionless first-, third-, and higher-order moments of $\phi-\psi$ as

$$\lambda_1 := \frac{2(n-1)}{e}\int g(\phi-\psi)dg,$$

$$\lambda_k := \frac{2(n-1)k!}{e(2k+n-4)!!}\int w_k(\phi-\psi)dg, \quad (3.3)$$

where

$$k! := 1\times 2\times 3\times\cdots\times k,$$

$$(2k+n-4)!! := \begin{cases} 2\times 4\times 6\times\cdots\times 2k-2 & \text{if } n=2, \\ 1\times 3\times 5\times\cdots\times 2k-1 & \text{if } n=3, \end{cases}$$

$$w_k := \sum_{l=0}^{[k/2]}(-1)^l\frac{(2k-2l+n-4)!!}{2^l l!(k-2l)!}g^{k-2l}.$$

Here $[k/2]$ means the largest integer less than or equal to $k/2$. It is evident from Eq. (2.4b) that $\lambda_1=0$. If now the dimensionless functions of v are introduced, defined by

$$\xi_1 := \frac{(n+1)v}{n+u}, \quad \xi_2 := \frac{(n+1)^2 u}{6(n+u)},$$

$$\xi_k := \frac{nk!(\sqrt{1-u})^{n+3}}{(n+u)(2k+n-4)!!}\int\frac{w_k}{(1-vg)^{n+1}}dg \quad (k\geq 3),$$

it can be checked by straightforward if tedious working that

$$\partial_t \lambda_k + \frac{1}{e}(\xi_k + \lambda_k)\partial_t e + \frac{d\xi_k}{dv}\partial_t v + \frac{k^2}{4k^2 - n + 2}\left[\partial_x \lambda_{k-1} + \frac{1}{e}(\xi_{k-1} + \lambda_{k-1})\partial_x e + \frac{d\xi_{k-1}}{dv}\partial_x v\right] + \partial_x \lambda_{k+1} + \frac{1}{e}(\xi_{k+1} + \lambda_{k+1})\partial_x e + \frac{d\xi_{k+1}}{dv}\partial_x v = -\varepsilon\eta(\xi_k + \lambda_k) - \frac{1}{\varepsilon}\lambda_k \quad (k \geq 2). \tag{3.4}$$

The system comprised of Eqs. (3.1) and (3.4) is formally equivalent to Eq. (2.6). Explaining more precisely the meaning of ξ_k, this quantity is a function of v such that when $|v| \ll 1$, then ξ_k is $O(v^k)$. The interpretation of $\xi_1+\lambda_1$, $\xi_2+\lambda_2$, and $\xi_k+\lambda_k$ $(k \geq 3)$ is that these are the moments of ϕ derived from Eq. (3.2) or Eq. (3.3) by replacing $\phi - \psi$ with ϕ.

B. Linearization

We now linearize Eqs. (3.1) and (3.4) about a constant equilibrium solution. This solution is given by

$$e = e' := \text{const} > 0, \quad v = v' := 0, \quad \lambda_k = \lambda_k' := 0.$$

The perturbation variables y_1 and y_2 are introduced as

$$e = e'(1 + y_1), \quad v = y_2.$$

Then linearization in the deviation from equilibrium yields the following system of equations for y_1, y_2, and λ_k:

$$\partial_t y_1 + \frac{n+1}{n}\partial_x y_2 = 0, \tag{3.5a}$$

$$\partial_t y_2 + \frac{1}{n+1}\partial_x y_1 + \frac{n}{n+1}\partial_x \lambda_2 = -\varepsilon\eta y_2, \tag{3.5b}$$

$$\partial_t \lambda_k + \frac{k^2}{4k^2 - n + 2}(\partial_x \lambda_{k-1} + \zeta_{k-1}\partial_x y) + \partial_x \lambda_{k+1} = -\left(\varepsilon\eta + \frac{1}{\varepsilon}\right)\lambda_k, \tag{3.5c}$$

where $k \geq 2$ and y, ζ_k are the column and row vectors defined by

$$y := [y_1, y_2]^T, \quad \zeta_k := \begin{cases} [0, (n+1)/n] & \text{if } k = 1, \\ [0,0] & \text{if } k \geq 2. \end{cases}$$

Here we recall that $\lambda_{k-1}=0$ when $k=2$.

In what follows consideration is first given to the Chapman-Enskog expansion of system (3.5) with $\eta=0$. Next the situation is described in which system (3.5) is taken with $\eta \neq 0$.

IV. CHAPMAN-ENSKOG EXPANSION

A. Overall structure

The types of solutions to system (3.5) obtained by the Chapman-Enskog method are a very special class of solutions, called normal solutions, in which the time and spatial dependence of $\lambda_k(t,x)$ appear implicitly through the perturbation variables and their spatial derivatives:

$$\lambda_k = \lambda_k(y, \partial_x y, \partial_x^2 y, \ldots, \partial_x^\infty y; \varepsilon) \quad (k \geq 2). \tag{4.1}$$

For the case $\eta=0$, assuming that the dependence on ε is as a power series, we postulate the following form of Eq. (4.1):

$$\lambda_k = \sum_{l=1}^{\infty} \varepsilon^l \lambda_{k|l} \partial_x^l y, \tag{4.2}$$

where $\lambda_{k|l}$ is a constant row vector,

$$\lambda_{k|l} := [\lambda_{k|l(1)}, \lambda_{k|l(2)}],$$

and $\lambda_{k|l}\partial_x^l y$ is a vector product defined by

$$\lambda_{k|l}\partial_x^l y := \lambda_{k|l(1)}\partial_x^l y_1 + \lambda_{k|l(2)}\partial_x^l y_2.$$

We also postulate that the equation for y takes the form

$$\partial_t y = -\sum_{l=0}^{\infty} \varepsilon^l Q_l \partial_x^{l+1} y, \tag{4.3}$$

where $Q_0, Q_1, \ldots, Q_\infty$ are the constant 2×2 matrices. A little algebra, aided by the substitution of

$$\lambda_2 = \sum_{l=1}^{\infty} \varepsilon^l \lambda_{2|l} \partial_x^l y$$

into the left-hand side of Eq. (3.5b), shows that

$$Q_0 = \begin{bmatrix} 0 & \frac{n+1}{n} \\ \frac{1}{n+1} & 0 \end{bmatrix}, \quad Q_l = \begin{bmatrix} 0 & 0 \\ \frac{n\lambda_{2|l(1)}}{n+1} & \frac{n\lambda_{2|l(2)}}{n+1} \end{bmatrix} \quad (l \geq 1). \tag{4.4}$$

Then, using the notation

$$\Lambda_{k|l} := \sum_{m=1}^{l} \lambda_{k|m} Q_{l-m},$$

the time derivative of λ_k can be evaluated; it turns out to be

$$\partial_t \lambda_k = -\sum_{l=1}^{\infty} \varepsilon^l \Lambda_{k|l} \partial_x^{l+1} y. \tag{4.5}$$

Proceeding formally, we substitute expansions (4.2), (4.3), and (4.5) into Eq. (3.5c) and equate coefficients of powers of ε. With the understanding that $\lambda_{1|l} := 0$, we obtain

$$\lambda_{2|1} = -\frac{4}{18 - n}[0, (n+1)/n], \quad \lambda_{k|1} = [0,0] \quad (k \geq 3), \tag{4.6a}$$

$$\lambda_{k|l+1} = \sum_{m=1}^{l} \lambda_{k|m} Q_{l-m} - \lambda_{k+1|l} - \frac{k^2}{4k^2 - n + 2}\lambda_{k-1|l} \quad (k \geq 2, l \geq 1). \tag{4.6b}$$

From Eq. (4.6) we get a set of recurrence relations for the

evaluation of $\lambda_{k|l}$. Equations (4.4) and (4.6a) can be used to find Q_1, and having $\lambda_{k|1}$, Q_1, and Eq. (4.6b), we can find $\lambda_{k|2}$, and so on. Also, each matrix Q_l can be successively computed from Eq. (4.4) by solving the recurrence equations. In Sec. IV B, we return to this derivation for more details. Clearly, in order to obtain a manageable system of equations, the infinite set $\{\lambda_{k|l}; k \geq 2, l \geq 1\}$ has to be truncated. If we truncate at pth order, we find the equation

$$\partial_t y = -\sum_{l=0}^{p} \varepsilon^l Q_l \partial_x^{l+1} y, \tag{4.7}$$

which is an approximation based on the knowledge of $\lambda_{2|1}, \lambda_{2|2}, \ldots, \lambda_{2|p}$. The equations with $p=1$, $p=2$, and $p=3$ correspond to truncation at first, second, and third order, respectively.

When $\eta \neq 0$, the situation is as follows. The power series expansions of λ_k and $\partial_t y$ are expressible as

$$\lambda_k = \sum_{l=1}^{\infty} \sum_{m=0}^{l} \varepsilon^l \lambda_{k|l}^{(m)} \partial_x^m y, \tag{4.8}$$

$$\partial_t y = -\varepsilon Q y - \sum_{l=0}^{\infty} \sum_{m=0}^{l} \varepsilon^l Q_{l|m} \partial_x^{m+1} y, \tag{4.9}$$

where

$$\lambda_{k|l}^{(m)} := [\lambda_{k|l(1)}^{(m)}, \lambda_{k|l(2)}^{(m)}], \quad Q := \begin{bmatrix} 0 & 0 \\ 0 & \eta \end{bmatrix},$$

$$Q_{0|0} := \begin{bmatrix} 0 & \dfrac{n+1}{n} \\ \dfrac{1}{n+1} & 0 \end{bmatrix}, \quad Q_{l|m} := \begin{bmatrix} 0 & 0 \\ \dfrac{n\lambda_{2|l(1)}^{(m)}}{n+1} & \dfrac{n\lambda_{2|l(2)}^{(m)}}{n+1} \end{bmatrix}, \tag{4.10}$$

with $l \geq 1$ and $m \leq l$. Working in a way analogous to the case $\eta = 0$, we first find that

$$\lambda_{k|l}^{(0)} = [0,0] \quad (k \geq 2, l \geq 1), \tag{4.11a}$$

$$\lambda_{2|1}^{(1)} = -\frac{4}{18-n}[0,(n+1)/n], \quad \lambda_{k|1}^{(1)} = [0,0] \quad (k \geq 3), \tag{4.11b}$$

$$\lambda_{2|l}^{(m)} = [0,0] \quad (1 \leq l \leq 3, m < l). \tag{4.11c}$$

However, in order to list the explicit form of further equations, the recurrence equations, we require the following formal extension to the definition of $\lambda_{k|l}^{(m)}$:

$$\lambda_{k|l}^{(m)} := [0,0] \quad (k \geq 2, m > l \geq 0),$$

$$\lambda_{1|l}^{(m)} := [0,0] \quad (l \geq 1, m \leq l),$$

$$Q_{l|m} := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (m > l \geq 0).$$

Note that this extension is consistent with Eqs. (4.8)–(4.11) and the relations below. The use of Eqs. (3.5c), (4.8), and (4.9) then yields

$$\lambda_{k|l+1}^{(m+1)} = \Lambda_{k|l}^{(m)} + \lambda_{k|l-1}^{(m+1)} Q - \eta \lambda_{k|l-1}^{(m+1)} - \lambda_{k+1|l}^{(m)} - \frac{k^2}{4k^2 - n + 2} \lambda_{k-1|l}^{(m)} \quad (k \geq 2, l \geq 1, m \leq l), \tag{4.12}$$

where

$$\Lambda_{k|l}^{(m)} := \sum_{r=1}^{l} \sum_{s=0}^{m} \lambda_{k|r}^{(s)} Q_{l-r|m-s}.$$

The equation for y can be approximated as

$$\partial_t y = -\varepsilon Q y - \sum_{l=0}^{p} \sum_{m=0}^{l} \varepsilon^l Q_{l|m} \partial_x^{m+1} y$$

and result (4.11c) introduces a great simplification into this approximation. It implies that if $p \leq 3$, then

$$\partial_t y = -\varepsilon Q y - \sum_{l=0}^{p} \varepsilon^l Q_{l|l} \partial_x^{l+1} y. \tag{4.13}$$

Here the matrices $Q_{l|l}$ have exactly the same meaning as in Eq. (4.4); i.e., these matrices satisfy the conditions $Q_{l|l} = Q_l$. Departures from the expansion of system (3.5) with $\eta = 0$ may therefore be expected when $p \geq 4$.

All the above assumes, either explicitly or implicitly, the following picture [22]. If ε is sufficiently small, then Eqs. (3.5a), (3.5b), and (3.5c) define a fast-slow system. This system has a manifold of slow motion $\mathfrak{M}_\varepsilon$ in the state space $\mathfrak{X}$. Each state corresponds to a unique point in $\mathfrak{X}$ and is represented by a sequence $\lambda := \{y, \lambda_k | k \geq 2\}$ composed of continuous and differentiable functions of x. There exists a function Ψ_ε whose graph is a slow manifold $\mathfrak{M}_\varepsilon$:

$$\Psi_\varepsilon(y) := \left\{ \sum_{l=1}^{\infty} \sum_{m=0}^{l} \varepsilon^l \lambda_{k|l}^{(m)} \partial_x^m y; k \geq 2 \right\}.$$

This manifold is locally invariant under the system dynamics and the dynamics on $\mathfrak{M}_\varepsilon$ is governed by Eq. (4.9).

B. Some explicit calculations

Equations (4.4) and (4.6) give

$$Q_0 = \begin{bmatrix} 0 & \frac{n+1}{n} \\ \frac{1}{n+1} & 0 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{2(n-1)}{n(n+2)} \end{bmatrix},$$

$$Q_2 = \begin{bmatrix} 0 & 0 \\ -\frac{1}{3(n+2)} & 0 \end{bmatrix}. \tag{4.14}$$

For Q_3, the solution is more complex; it reads

$$Q_3 = \mathfrak{d} Q_1, \tag{4.15}$$

where

$$\mathfrak{d} := \frac{2(13n+46)}{(7n-6)(38-n)} > 0. \tag{4.16}$$

To evaluate $Q_{0|0}$, $Q_{1|1}$, $Q_{2|2}$, and $Q_{3|3}$ it need only be recalled, as found at the end of Sec. IV A, that $Q_{l|l}=Q_l$ for $l \leq 3$. In other words, to the extent the equation for y is calculated up to order 3 in ε, see Eq. (4.13) with $p \leq 3$, all the differences between Eqs. (4.6) and (4.12) can be ignored:

$$\lambda_{k|l-1}^{(m+1)} Q - \eta \lambda_{k|l-1}^{(m+1)} = 0 \quad (\eta \geq 0, l = 1,2).$$

Nevertheless, it is important to stress that the matrix Q is not a zero matrix when $\eta \neq 0$. Consequently, even if we assume that $p \leq 3$, Eq. (4.13) may differ from Eq. (4.7).

C. Entropy

Let h_0 be the 2×2 symmetric positive-definite matrix defined in such a way that the matrix $h_0 Q_0$ is symmetric and the matrix $-h_0 Q_1$ is symmetric and positive semidefinite:

$$\alpha_0 := h_0 Q_0 = (h_0 Q_0)^T, \quad \alpha_1 := -2 h_0 Q_1 = -2 (h_0 Q_1)^T \geq 0. \tag{4.17}$$

The solution of Eq. (4.17) for h_0 is unique up to multiplication by a positive constant and from Eq. (4.14) we find h_0 to be

$$h_0 = \begin{bmatrix} \frac{n^2}{(n+1)^2} & 0 \\ 0 & n \end{bmatrix}. \tag{4.18}$$

Then $h_0 > 0$ and by Eq. (4.17),

$$\alpha_0 = \begin{bmatrix} 0 & \frac{n}{n+1} \\ \frac{n}{n+1} & 0 \end{bmatrix}, \quad \alpha_1 = \begin{bmatrix} 0 & 0 \\ 0 & \frac{4(n-1)}{n+2} \end{bmatrix}. \tag{4.19}$$

The matrix α_1 is positive semidefinite, but not positive definite. In this sense, the matrix α_1 is degenerate and leads to a degenerate expression for the entropy production Σ as defined below.

In an attempt to formulate the H theorem for the third-order equation

$$\partial_t y = -Q_0 \partial_x y - \varepsilon (Q y + Q_1 \partial_x^2 y) - \varepsilon^2 Q_2 \partial^3 y - \varepsilon^3 Q_3 \partial_x^4 y, \tag{4.20}$$

where $Q, Q_0, \ldots, Q_3$ are defined by Eqs. (4.10), (4.14), and (4.15), it is natural to aim for simplicity by introducing the matrices

$$\alpha_2 := h_0 Q_2 + (h_0 Q_2)^T, \quad \beta_2 := h_0 Q_2 - (h_0 Q_2)^T.$$

Explicitly, we obtain

$$\alpha_2 = \begin{bmatrix} 0 & -\frac{n}{3(n+2)} \\ -\frac{n}{3(n+2)} & 0 \end{bmatrix},$$

$$\beta_2 = \begin{bmatrix} 0 & \frac{n}{3(n+2)} \\ -\frac{n}{3(n+2)} & 0 \end{bmatrix}.$$

It is also convenient to introduce the following symmetric positive-semidefinite matrix:

$$h_2 := \begin{bmatrix} \frac{n^2}{3(n+1)(n+2)} & 0 \\ 0 & 0 \end{bmatrix}.$$

Now, define the gradient-dependent entropy by

$$\mathcal{H} := y^T h_0 y + \varepsilon^2 (\partial_x y)^T h_2 \partial_x y. \tag{4.21}$$

This quantity is a non-negative quadratic form that vanishes if and only if

$$y = [0,0]^T, \quad \partial_x y = [0, \partial_x y_2]^T.$$

Differentiating $\mathcal{H}$ with respect to time and using Eq. (4.20) yields

$$\partial_t \mathcal{H} + \partial_x \Phi = \Sigma, \tag{4.22}$$

where

$$\Phi := y^T \alpha_0 y - \varepsilon y^T \alpha_1 \partial_x y + \varepsilon^2 y^T (\alpha_2 + \beta_2) \partial_x^2 y - \varepsilon^2 (\partial_x y)^T \alpha_2 \partial_x y - \varepsilon^3 y^T \alpha_3 \partial_x^3 y + \varepsilon^3 (\partial_x y)^T \alpha_3 \partial_x^2 y, \tag{4.23a}$$

$$\Sigma := -\varepsilon y^T Q y - \varepsilon (\partial_x y)^T \alpha_1 \partial_x y + \varepsilon^3 (\partial_x^2 y)^T \alpha_3 \partial_x^2 y, \tag{4.23b}$$

with α_3 given by

$$\alpha_3 := -h_0 Q_3 - (h_0 Q_3)^T = \mathfrak{d} \alpha_1. \tag{4.24}$$

It is evident from Eqs. (4.16), (4.17), and (4.24) that $\alpha_3 \geq 0$.

From a physical point of view, it may seem that the entropylike quantity $\mathcal{H}$ is postulated *ad hoc*. However, at least in the case of a first-order approximation to Eq. (4.20),

$$\partial_t y = -Q_0 \partial_x y - \varepsilon (Q y + Q_1 \partial_x^2 y),$$

it is possible to relate $\mathcal{H}$ directly to the second differential of the true entropy density. For more details, see Banach and Larecki ([5], Sec. VI). Let

$$V := \int_{\mathbb{R}} \mathcal{H}(t,x)dx, \quad P := \int_{\mathbb{R}} \Sigma(t,x)dx.$$

Assuming that $y(t,x)$ decays fast enough if $|x| \to \infty$, we arrive at the equation $\dot{V}=P$. The lack of evidence that $\dot{V} \leq 0$ follows from the fact that $\alpha_3 \geq 0$.

These are our previous results [5] and we briefly summarize them as follows. There exist the 2×2 matrices h_0 and h_2 such that

$$h_0 = (h_0)^T > 0, \quad h_2 = (h_2)^T \geq 0, \quad \alpha_0 = (\alpha_0)^T,$$

$$\alpha_1 = (\alpha_1)^T \geq 0.$$

For this reason, truncation at order ε^2 gives the equations which satisfy the H theorem and possess the property of yielding stable constant solutions. However, since the matrix α_3 is not negative semidefinite, the equations corresponding to the next order of approximation exhibit instability to small wavelength disturbances and are inconsistent with thermodynamics in the sense that the nonpositivity of Σ is not automatically guaranteed for all possible conditions of the phonon gas. As an illustration, the substitution of Eq. (4.24) into Eq. (4.23b) and the use of Eqs. (4.10), (4.19), and (4.24) lead to

$$\Sigma = -\varepsilon \eta (y_2)^2 - \frac{4\varepsilon(n-1)}{n+2}(\partial_x y_2)^2 + \frac{4\varepsilon^3 \mathfrak{d}(n-1)}{n+2}(\partial_x^2 y_2)^2,$$

then Eq. (4.16) implies that $\Sigma > 0$ if, e.g., $y_2 = \partial_x y_2 = 0$ and $\partial_x^2 y_2 \neq 0$.

V. VARIABLE TRANSFORMATION APPROACH

A. Preliminaries

In Sec. IV A, the state variable λ is represented in the standard basis of $\mathfrak{X}$:

$$\lambda := \{y, \lambda_k | k \geq 2\}.$$

In this section, we consider a different coordinate system in $\mathfrak{X}$,

$$\lambda := \{z, \lambda_k | k \geq 2\},$$

and derive the equation for z that governs the dynamics on $\mathfrak{M}_\varepsilon$. We note that the idea of special transformation for the proper fast-slow separation is not new in kinetics. Similar ideas were invented for chemical kinetics by Lam and Goussis [23] and developed further by many researches [24]. They call this method the "computational singular perturbation." For our purposes, it suffices to implement a reduced (one-step) version of the method. The transformation is the same at every point of $\mathfrak{X}$. Moreover, this transformation is simple in that it changes only the slow variables.

Consider the system

$$\partial_t y = -\sum_{l=0}^{p} \varepsilon^l Q_l \partial_x^{l+1} y$$

derived in Sec. IV A under the assumption that $Q=0$. The essence of the technique proposed here consists in constructing a linear differential transformation $(y, \varepsilon) \mapsto z$ analytic in ε at $\varepsilon=0$ so as to achieve in the transformed system

$$\partial_t z = -\sum_{l=0}^{p} \varepsilon^l R_l \partial_x^{l+1} z \tag{5.1}$$

specific requirements (stability of the zero solution, existence of a local H theorem, elimination of the degenerate expression for the entropy production, and so on). We propose to build the transformation in the power series form

$$z = y + \sum_{l=1}^{\infty} \varepsilon^l Y_l \partial_x^l y, \tag{5.2}$$

where $Y_1, Y_2, \ldots, Y_\infty$ are the constant 2×2 matrices. The inverse transformation is given by

$$y = z + \sum_{l=1}^{\infty} \varepsilon^l Z_l \partial_x^l z, \tag{5.3}$$

where

$$Z_1 = -Y_1, \quad Z_l = -Y_l - \sum_{m=1}^{l-1} Y_m Z_{l-m} \quad (l \geq 2).$$

The scheme is basically a recursive one; it is based on the principle that, with respect to transformation (5.2), system (4.3) is equivalent to the system

$$\partial_t z = -\sum_{l=0}^{\infty} \varepsilon^l R_l \partial_x^{l+1} z \tag{5.4}$$

derived from the equation

$$\partial_t z = \partial_t y + \sum_{l=1}^{\infty} \varepsilon^l Y_l \partial_x^l \partial_t y$$

by using Eqs. (4.3) and (5.3). The matrices R_l are therefore

$$R_0 = Q_0, \quad R_1 = Q_1 + Y_1 Q_0 - Q_0 Y_1,$$

$$R_l = Q_l + Y_l Q_0 - Q_0 Y_l + \sum_{m=1}^{l-1} (Y_m Q_{l-m} - Q_0 Y_m Z_{l-m})$$

$$+ \sum_{m=1}^{l-1} \left(Q_m + \sum_{r=1}^{m} Y_r Q_{m-r} \right) Z_{l-m} \quad (l \geq 2). \tag{5.5}$$

In this way, the complete specification of system (5.4) is obtained, which enables access to further development opportunities. The transformation defined by Eq. (5.2) is canonical, in the sense that system (5.4) has the same form as system (4.3).

If we repeat the construction for system (4.9) with $Q \neq 0$, then we find that the transformation does not satisfy the criterion of canonicity. Consequently, insofar as the explicit description of a theory of transformations is concerned, we restrict ourselves to the case $Q=0$ and do not present the sequence of operations one has to perform when $Q \neq 0$. Clearly, under this restriction, system (4.9) reduces to the system comprised of Eqs. (4.3) and (4.4).

We end this subsection with the remark explaining what does truncation in new variables mean for original series: if one transforms y to z, then truncates the series for time derivative for z, and returns back to y, then how will the equations for y look like? For mathematical convenience, we assume that our perturbation variables are harmonic in space and time. So we have, effectively,

$$y = y_k \exp(i\omega t - ikx), \quad z = z_k \exp(i\omega t - ikx), \tag{5.6}$$

where ω is the wave frequency, y_k is the complex amplitude of y, and z_k is the complex amplitude of z. The wave number k will be regarded as real throughout. Let I be the unit matrix and let

$$M := I + \sum_{l=1}^{p} \varepsilon^l (-ik)^l Z_l.$$

Substitution from Eq. (5.6) into

$$y = z + \sum_{l=1}^{p} \varepsilon^l Z_l \partial_x^l z$$

gives

$$y_k = M z_k \tag{5.7}$$

and the substitution of Eq. (5.6) into Eq. (5.1) gives

$$i\omega z_k = -\sum_{l=0}^{p} \varepsilon^l (-ik)^{l+1} R_l z_k. \tag{5.8}$$

Equations (5.7) and (5.8) combine into

$$i\omega y_k = -\sum_{l=0}^{p} \varepsilon^l (-ik)^{l+1} M R_l M^{-1} y_k, \tag{5.9}$$

which puts in evidence the fact that we do not arrive at Eq. (4.7) when $Y_l \neq 0$ and $p < \infty$.

In order to illustrate these remarks with a specific example, we consider Eq. (5.9) with $p=1$. Defining Y_1 by Eq. (5.12), we find, after a little algebra, that

$$\omega y_{k(1)} - \frac{n+1}{n} k y_{k(2)} = 0,$$

$$\omega y_{k(2)} - \frac{1}{n+1} k y_{k(1)} - \frac{2(n-1)i}{n(n+2)} \varepsilon k^2 y_{k(2)} - \frac{2(n-1)(1-d)d}{3(n+2)^2} \varepsilon^2 k^3 y_{k(1)} = 0,$$

where d is the numerical parameter and $y_{k(1)}$ and $y_{k(2)}$ are the components of y_k. The equation for y is therefore

$$\partial_t y = -Q_0 \partial_x y - \varepsilon Q_1 \partial_x^2 y - \varepsilon^2 \theta_2 Q_2 \partial_x^3 y,$$

where

$$\theta_2 = \frac{2(n-1)(1-d)d}{n+2}.$$

This equation is stable as long as $0 \leq d \leq 1$. So we have found a one-parameter family of equations of the second-order theory with linear stability.

B. H theorem

The second-order approximation to Eq. (5.4) results in the following equation for z:

$$\partial_t z = -R_0 \partial_x z - \varepsilon R_1 \partial_x^2 z - \varepsilon^2 R_2 \partial_x^3 z, \tag{5.10}$$

where R_0 and R_1 are given by Eq. (5.5) and R_2 is given by

$$R_2 = Q_2 + Y_2 Q_0 - Q_0 Y_2 - (Y_1 Q_0 - Q_0 Y_1) Y_1 + Y_1 Q_1 - Q_1 Y_1.$$

Suppose h_0 takes form (4.18). Also, define Y_1 and Y_2 in such a way that

$$\alpha_1 := -2h_0 R_1 = -2(h_0 R_1)^T > 0, \quad \beta_2 := h_0 R_2 - (h_0 R_2)^T = 0. \tag{5.11}$$

For the sake of concreteness, we define Y_1 and Y_2 by

$$Y_1 = \begin{bmatrix} 0 & 0 \\ \frac{nd}{3(n+2)} & 0 \end{bmatrix}, \quad Y_2 = \begin{bmatrix} \frac{(n-1)\mu}{n(n+2)} & 0 \\ 0 & 0 \end{bmatrix}, \tag{5.12}$$

where

$$0 < d < 1, \quad \mu = -1 + \frac{2(n-1)(1-d)d}{n+2}. \tag{5.13}$$

This yields

$$\alpha_1 = \left[\begin{array}{c|c} \frac{2n^2 d}{3(n+1)(n+2)} & 0 \\ \hline 0 & \frac{4(n-1)(1-d)}{n+2} \end{array}\right]$$

and then the matrix α_1 is positive definite—exactly as it should be. Because of Eq. (5.13), the uncertainty in the definition of Eq. (5.10) is reduced to the numerical parameter d in Eq. (5.12). The entries of R_1 and R_2 become functions of d and we obtain

$$R_0 = Q_0, \quad R_1 = \hat{R}_1(d), \quad R_2 = \hat{R}_2(d). \tag{5.14}$$

Equation (5.10) with Eq. (5.14) seems to be the most natural replacement for Eq. (4.7). Its nonuniqueness is unavoidable unless some additional restrictions are introduced to specify d. The natural defining relation for d is the condition that the matrix α_1 is the unit matrix multiplied by a constant. This evidently implies

$$0 < d = \frac{(n+1)^2}{1+3n+n^2} < 1$$

and then the constant is positive.

Using Eq. (5.11) and observing that

$$R_0 = Q_0, \quad h_0 R_0 = (h_0 R_0)^T, \tag{5.15}$$

the equation analogous to Eq. (4.22) holds with Eqs. (4.21) and (4.23) replaced by

$$\mathcal{H} := z^T h_0 z,$$

$$\Phi := z^T \alpha_0 z - \varepsilon z^T \alpha_1 \partial_x z + \varepsilon^2 z^T \alpha_2 \partial_x^2 z - \frac{1}{2}\varepsilon^2 (\partial_x z)^T \alpha_2 \partial_x z, \tag{5.16a}$$

$$\Sigma := -\varepsilon (\partial_x z)^T \alpha_1 \partial_x z, \tag{5.16b}$$

where

$$\alpha_0 := h_0 R_0, \quad \alpha_2 := 2 h_0 R_2.$$

Explicitly, differentiating Eq. (5.16a) with respect to time, making use of Eqs. (5.10) and (5.11), and noting Eq. (5.15), we obtain the balance law for $\mathcal{H}$—the second law—as

$$\partial_t \mathcal{H} + \partial_x \Phi = \Sigma. \tag{5.17}$$

The important thing to keep in mind is that $\mathcal{H}$ is a local function of z and that Σ satisfies the inequality $\Sigma \leq 0$ with equality if and only if $\partial_x z = 0$. As a matter of fact, the implications of the simple Eqs. (5.16a) and (5.16b) can be put most vividly in the common case for which $z(t,x) \neq 0$. For then Eq. (5.17) gives

$$\frac{d}{dt}\int_{\mathbb{R}} \mathcal{H}(t,x)dx < 0$$

and it follows at once that the zero solution of Eq. (5.10) is stable.

As a final way of looking at the same things, we follow the procedure, already discussed in Ref. [5], in which the vector field z is assumed to be harmonic in space and time, and so proportional to $\exp(i\omega t - ikx)$. However, we must begin by defining the most general class of transformations that could arise realistically in our analysis of Eqs. (5.18) and (5.19).

C. Problem of stability

In order to gain a deeper understanding of the variable transformation approach, we insert the two parameters θ_2 and θ_3 into Eq. (4.20),

$$\partial_t y = -Q_0 \partial_x y - \varepsilon Q_1 \partial_x^2 y - \varepsilon^2 \theta_2 Q_2 \partial_x^3 y - \varepsilon^3 \theta_3 Q_3 \partial_x^4 y, \tag{5.18}$$

setting $Q=0$ and assuming that Q_0, Q_1, Q_2, Q_3 are given by Eqs. (4.14) and (4.15). The domain of stability For Eq. (5.18) is as follows:

$$\theta_2 \geq 0, \quad \theta_3 \leq 0.$$

If $\theta_2 < 0$ or $\theta_3 > 0$, since then either $\theta_2 h_2$ or $-\theta_3 \alpha_3$ is negative semidefinite, this equation possesses an unstable zero solution and appears to be inconsistent with thermodynamics.

After making the change in variable from y to z and neglecting the terms of order higher than cubic in ε, we find

$$\partial_t z = -R_0 \partial_x z - \varepsilon R_1 \partial_x^2 z - \varepsilon^2 R_2 \partial_x^3 z - \varepsilon^3 R_3 \partial_x^4 z, \tag{5.19}$$

where

$$R_0 = Q_0, \quad R_1 = Q_1 + Y_1 Q_0 - Q_0 Y_1,$$

$$R_2 = \theta_2 Q_2 + Y_2 Q_0 - Q_0 Y_2 - (Y_1 Q_0 - Q_0 Y_1) Y_1 + Y_1 Q_1 - Q_1 Y_1,$$

$$\begin{aligned} R_3 = {} & \theta_3 Q_3 + Y_3 Q_0 - Q_0 Y_3 - (Y_1 Q_0 - Q_0 Y_1)(Y_2 - Y_1^2) + Y_2 Q_1 \\ & - Q_1 Y_2 + \theta_2 (Y_1 Q_2 - Q_2 Y_1) - (Y_1 Q_1 - Q_1 Y_1) Y_1 \\ & - (Y_2 Q_0 - Q_0 Y_2) Y_1. \end{aligned} \tag{5.20}$$

Assume that $z(t,x)$ decays fast enough if $|x| \to \infty$. Using the notation

$$\alpha_1 := -h_0 R_1 - (h_0 R_1)^T, \quad \beta_2 := h_0 R_2 - (h_0 R_2)^T,$$

$$\alpha_3 := -h_0 R_3 - (h_0 R_3)^T, \tag{5.21}$$

where h_0 takes form (4.18), and setting

$$\mathcal{H} := z^T h_0 z, \quad \Sigma := -\varepsilon (\partial_x z)^T \alpha_1 \partial_x z + \varepsilon^3 (\partial_x^2 z)^T \alpha_3 \partial_x^2 z,$$

we verify that Eq. (5.19) is stable and satisfies the H theorem

$$\frac{d}{dt}\int_{\mathbb{R}} \mathcal{H}(t,x)dx = \int_{\mathbb{R}} \Sigma(t,x)dx \leq 0$$

if and only if $\alpha_3 \leq 0$ and

$$\alpha_1 \geq 0, \quad \beta_2 = 0. \tag{5.22}$$

The most general solution of Eq. (5.22) is

$$Y_1 = \begin{bmatrix} \frac{(n+1)a}{n+2} & \frac{3(n+1)^2\kappa}{n+2} \\ \frac{n(d+9\kappa)}{3(n+2)} & \frac{(n+1)b}{n+2} \end{bmatrix},$$

$$Y_2 = \begin{bmatrix} \frac{(n-1)\mu}{n(n+2)} & r \\ s & \frac{(n-1)\nu}{n(n+2)} \end{bmatrix}, \tag{5.23}$$

where

$$0 \leq d \leq 1, \quad (a,b,\kappa,r,s,\nu) \in \mathbb{R}^6, \tag{5.24a}$$

$$\mu = -\theta_2 + \nu + \frac{3}{5}(n+1)(n+3)\kappa \tag{5.24b}$$

$$+ \frac{2(n-1)(1-d)d}{n+2} + \frac{3(n+1)(a^2 - b^2 - 2n\kappa d)}{n+2}, \tag{5.24c}$$

so that the problem of finding the stable and entropy-consistent equation reduces to the problem of finding the matrix Y_3 such that $\alpha_3 \leq 0$. We can assume without any loss of generality that

$$Y_3 = \left[\begin{array}{c|c} \kappa_1 & (n+1)(\delta+\kappa_2) \\ \hline n\kappa_2/(n+2) & \nu_1+\kappa_1 \end{array}\right], \tag{5.25}$$

where $(\delta, \nu_1, \kappa_1, \kappa_2)$ are arbitrary constants. Let $\alpha_{3(ij)}$ be the (i,j) entry of α_3. Then, from Eqs. (5.20) and (5.21),

$$\alpha_3 = \begin{bmatrix} \alpha_{3(11)} & \alpha_{3(12)} \\ \alpha_{3(12)} & \dfrac{4(n-1)}{n+2}\mathfrak{d}\theta_3 - \dfrac{(n+1)^2}{n}\alpha_{3(11)} \end{bmatrix}.$$

When $\theta_3 > 0$, since $\mathfrak{d} > 0$, the condition $\alpha_3 \le 0$ gives nothing; but $\theta_3 \le 0$, Eq. (5.12),

$$0 \le d \le 1, \quad \mu = -\theta_2 + \frac{2(n-1)(1-d)d}{n+2}, \tag{5.26}$$

and Eq. (5.25) lead to

$$\alpha_3 = \begin{bmatrix} -\dfrac{2n^2\delta}{(n+1)^2} & 0 \\ 0 & \dfrac{4(n-1)\mathfrak{d}\theta_3}{n+2} + 2n\delta \end{bmatrix}$$

and

$$0 \le \delta \le \frac{2(n-1)\mathfrak{d}|\theta_3|}{n(n+2)}. \tag{5.27}$$

We can say that if $\theta_2 < 0$, there exists a possibility for restoring stability, but only for $\theta_3 \le 0$. As previously, therefore, the rest state is unstable when $\theta_3 > 0$.

We continue and conclude this section by giving some indication of the way in which similar results can also be obtained from the standard Fourier analysis of Eq. (5.19). In order to deal with plane waves, we make the perturbation vector z harmonic in space and time, and represent it by a complex amplitude z_k with factor $\exp(i\omega t - ikx)$ assumed, as usual. Then Eq. (5.19) can be written as

$$(\omega I - X - iY)z_k = 0,$$

where

$$X := k(R_0 - \varepsilon^2 k^2 R_2), \quad Y := -\varepsilon k^2 (R_1 - \varepsilon^2 k^2 R_3).$$

Let the dispersion relation

$$\det(\omega I - X - iY) = 0$$

be regarded as an equation for ω in terms of $k \in \mathbb{R}$. We obtain the quadratic equation

$$\omega^2 - (A + iB)\omega + C + iD = 0$$

and the minimum we need to know is that (A, B, C, D) are the real functions of k and that B is related to the traces of R_1 and R_3 by

$$B = -\varepsilon k^2 \operatorname{Tr}(R_1) + \varepsilon^3 k^4 \operatorname{Tr}(R_3). \tag{5.28}$$

Since the trace of the matrix R_3 is invariant under a set of all transformations,

$$\operatorname{Tr}(R_3) = \operatorname{Tr}(Q_3) = -\frac{2(n-1)}{n(n+2)}\mathfrak{d}\theta_3,$$

Eq. (5.28) simplifies to

$$B = -\varepsilon k^2 \operatorname{Tr}(R_1) - \frac{2(n-1)}{n(n+2)}\varepsilon^3 k^4 \mathfrak{d}\theta_3. \tag{5.29}$$

The imaginary part of ω appears in the form

$$\operatorname{Im}(\omega) = \frac{1}{2}(B \pm \sqrt{\sqrt{\mathbb{A}^2 + \mathbb{B}^2} - \mathbb{A}}), \tag{5.30}$$

where

$$\mathbb{A} := \frac{1}{2}(A^2 - B^2 - 4C), \quad \mathbb{B} := AB - 2D.$$

Stability requires $\operatorname{Im}(\omega) \ge 0$, i.e.,

$$B \ge 0, \quad B \ge \sqrt{\sqrt{\mathbb{A}^2 + \mathbb{B}^2} - \mathbb{A}}.$$

If $|k| \to \infty$, the right-hand side of Eq. (5.29) is given essentially by the second term. This term is negative or positive according as θ_3 is positive or negative, thereby implying that the rest state is unstable when $\theta_3 > 0$. For $\theta_3 \le 0$, on the other hand, using

$$\operatorname{Tr}(R_1) = \operatorname{Tr}(Q_1) = -\frac{2(n-1)}{n(n+2)},$$

we have $B \ge 0$. In this case, the H theorem for Eq. (5.19) provides a hint on what to do next. This hint is just a way of suggesting what the most general class of entropy-consistent transformations is.

We first consider the second-order approximation to Eq. (5.19),

$$\theta_3 = 0, \quad B = \frac{2(n-1)}{n(n+2)}\varepsilon k^2,$$

assuming for simplicity that $\theta_2 = \pm 1$. With the help of Eqs. (5.23) and (5.24), and

$$a = b = r = s = \kappa = \nu = 0,$$

the calculation of $\mathbb{A}$ and $\mathbb{B}$ is relatively easy and when it is completed Eq. (5.30) is

$$\operatorname{Im}(\omega) = \frac{1}{2}\left[B \pm \sqrt{\left|Ek^2 - \frac{1}{2}B^2\right| - \left(Ek^2 - \frac{1}{2}B^2\right)}\right],$$

where

$$E = 1 + \frac{1}{4}\theta_2\varepsilon^2 k^2 + \frac{1}{256}(2\theta_2 - d + d^2)^2 \varepsilon^4 k^4, \tag{5.31a}$$

if $n = 2$ and

$$E = \frac{2}{3} + \frac{8}{45}\theta_2\varepsilon^2 k^2 + \frac{8}{16\,875}(5\theta_2 - 4d + 4d^2)^2 \varepsilon^4 k^4, \tag{5.31b}$$

if $n = 3$. Let $0 \le d \le 1$. Writing Eq. (5.31) in the form

$$E = \mathcal{A}\varepsilon^4 k^4 + \theta_2 \mathcal{B}\varepsilon^2 k^2 + \mathcal{C},$$

we verify that $\mathcal{A}>0$, $\mathcal{B}>0$, and $\mathcal{C}>0$. We also verify that if $\theta_2=-1$, then $\mathcal{B}^2-4\mathcal{A}\mathcal{C}<0$. Ultimately, therefore, $E>0$ and this implies $\mathrm{Im}(\omega)\geq 0$. Equation (5.18) with $\theta_2=-1$ and $\theta_3=0$ is unstable. The transformed equation is stable regardless of the sign of θ_2.

Finally, we observe that a very similar analysis applies to the case $\theta_3<0$. Without entering into detailed calculations and by relying on structural arguments alone, it will be seen from Eqs. (5.12) and (5.25)–(5.27), and

$$\nu_1 = \kappa_1 = \kappa_2 = 0,$$

that

$$\mathbb{B} = 0, \quad \mathbb{A} + \frac{1}{2}B^2 \geq 0.$$

By Eq. (5.30) we then have $\mathrm{Im}(\omega)\geq 0$. Thus, the zero solution of Eq. (5.19) is stable when $\theta_3<0$.

VI. EXISTENCE AND ASYMPTOTIC STABILITY

We consider the following initial-boundary value problem:

$$z_{1,t} = -b_1 z_{2,x} + a_1 z_{1,xx} \text{ in } (a,b)\times(0,T), \tag{6.1a}$$

$$z_{2,t} = -b_2 z_{1,x} + a_2 z_{2,xx} \text{ in } (a,b)\times(0,T), \tag{6.1b}$$

$$z_i|_{x\in\{a,b\}} = 0, \quad i=1,2, \text{ on } (0,T), \tag{6.1c}$$

$$z_i|_{t=0} = z_{i0}, \quad i=1,2, \text{ in } (a,b), \tag{6.1d}$$

where $(a,b)\subset\mathbb{R}$ is an interval and a_1, a_2, b_1, and b_2 are positive constants. With the choice

$$a_1 = \frac{\varepsilon(n+1)d}{3(n+2)}, \quad a_2 = \frac{\varepsilon(n+1)(1-d)}{3(n+2)}, \quad 0<d<1, \tag{6.2a}$$

$$b_1 = \frac{n+1}{n}, \quad b_2 = \frac{1}{n+1}, \tag{6.2b}$$

provided that Y_1 is defined by Eq. (5.12), this problem is equivalent to the initial-boundary value problem for Eq. (5.1) with $p=1$. Introducing the new variables

$$u_1 = \sqrt{b_2}z_1, \quad u_2 = \sqrt{b_1}z_2,$$

problem (6.1) takes the form

$$u_{1,t} = -\sqrt{b_1 b_2}u_{2,x} + a_1 u_{1,xx} \text{ in } (a,b)\times(0,T), \tag{6.3a}$$

$$u_{2,t} = -\sqrt{b_1 b_2}u_{1,x} + a_2 u_{2,xx} \text{ in } (a,b)\times(0,T), \tag{6.3b}$$

$$u_i|_{x\in\{a,b\}} = 0, \quad i=1,2, \text{ on } (0,T), \tag{6.3c}$$

$$u_i|_{t=0} = u_{i0}, \quad i=1,2, \text{ in } (a,b), \tag{6.3d}$$

where $u_{10}=\sqrt{b_2}z_{10}$ and $u_{20}=\sqrt{b_1}z_{20}$.

Lemma 6.1. Assume that $u=(u_1,u_2)$ is a sufficiently smooth solution to Eq. (6.3). Then the decay estimate

$$\int_a^b (u_1^2+u_2^2)dx \leq e^{-2c_1 a_* t}\int_a^b (u_{10}^2+u_{20}^2)dx \tag{6.4}$$

holds, where c_1 is the constant from the Poincaré inequality and $a_*=\min\{a_1,a_2\}$.

Proof. Multiplying Eq. (6.3a) by u_1, Eq. (6.3b) by u_2, and adding the results yield

$$\frac{1}{2}\frac{d}{dt}(u_1^2+u_2^2) = -\sqrt{b_1 b_2}(u_1 u_2)_{,x} + a_1 u_{1,xx}u_1 + a_2 u_{2,xx}u_2. \tag{6.5}$$

Let

$$\mathcal{H} = \frac{1}{2}(u_1^2+u_2^2), \quad \Phi = \sqrt{b_1 b_2}u_1 u_2 - a_1 u_{1,x}u_1 - a_2 u_{2,x}u_2,$$

$$\Sigma = -(a_1 u_{1,x}^2 + a_2 u_{2,x}^2).$$

Then Eq. (6.5) assumes the form of conservation law

$$\mathcal{H}_{,t} + \Phi_{,x} = \Sigma, \tag{6.6}$$

with energy $\mathcal{H}$, flux Φ, and dissipation Σ. Integrating Eq. (6.6) over (a,b) and using the boundary conditions give

$$\frac{1}{2}\frac{d}{dt}\int_a^b (u_1^2+u_2^2)dx = -\int_a^b (a_1 u_{1,x}^2 + a_2 u_{2,x}^2)dx. \tag{6.7}$$

In view of the Poincaré inequality

$$c_1\int_a^b u_i^2 dx \leq \int_a^b u_{i,x}^2 dx, \quad i=1,2,$$

and for $a_*=\min\{a_1,a_2\}$, Eq. (6.7) implies

$$\frac{1}{2}\frac{d}{dt}\int_a^b (u_1^2+u_2^2)dx \leq -c_1 a_*\int_a^b (u_1^2+u_2^2)dx. \tag{6.8}$$

Integrating Eq. (6.8) with respect to time and employing the initial conditions yield Eq. (6.4). This concludes the proof. □

Integrating Eq. (6.7) with respect to time, we obtain the energy estimate

$$\frac{1}{2}\int_a^b (u_1^2+u_2^2)dx + a_*\int_0^t dt'\int_a^b (u_{1,x}^2+u_{2,x}^2)dx \leq \frac{1}{2}\int_a^b (u_{10}^2+u_{20}^2)dx, \quad t\in(0,T). \tag{6.9}$$

Inequality (6.9) suggests that existence of weak solutions can be proved.

Definition 6.2. By a weak solution to problem (6.3) we mean the function $u=(u_1,u_2)$ satisfying the integral identity

$$\int_0^T dt \int_a^b (u_1\psi_{1,t} + u_2\psi_{2,t})dx - \int_0^T dt \int_a^b (a_1 u_{1,x}\psi_{1,x} + a_2 u_{2,x}\psi_{2,x})dx - \sqrt{b_1 b_2}\int_0^T dt \int_a^b (u_{2,x}\psi_1 + u_{1,x}\psi_2)dx + \int_a^b (u_{10}\psi_{10} + u_{20}\psi_{20})dx = 0, \tag{6.10}$$

which holds for any $\psi=(\psi_1,\psi_2)\in W_2^{1,1}[(a,b)\times(0,T)]$ such that $\psi|_{x\in\{a,b\}}=0$, $\psi|_{t=T}=0$, where $\psi_{i0}=\psi_i|_{t=0}$, $i=1,2$.

Remark 6.3. For the reader's convenience, the space $W_2^{1,1}[(a,b)\times(0,T)]$ and other spaces which we use here are defined in the Appendix. See also Refs. [25,26].

Lemma 6.4. Assume that $u_0=(u_{10},u_{20})\in L_2(a,b)$. Then there exists a weak solution to problem (6.3) such that $u=(u_1,u_2)\in V_2^0[(a,b)\times(0,T)]$ and the estimate

$$\|u\|_{V_2^0[(a,b)\times(0,T)]} \le c\|u_0\|_{L_2(a,b)} \tag{6.11}$$

is valid.

Proof. We prove the existence by the Galerkin method, so we are looking for approximate solutions in the form

$$u_i^n = \sum_{k=1}^n u_{ik}^n(t)e_k(x), \quad i=1,2, \tag{6.12}$$

where $\{e_k\}$ is a basis in $H_0^1(a,b)$. Then $u_{ik}^n(t)$ are solutions to the following system of ordinary differential equations:

$$\frac{d}{dt}u_{1k}^n = -\sqrt{b_1 b_2}\sum_{l=1}^n u_{2l}^n \int_a^b e_{l,x}e_k dx - a_1\sum_{l=1}^n u_{1l}^n \int_a^b e_{l,x}e_{k,x}dx, \tag{6.13a}$$

$$\frac{d}{dt}u_{2k}^n = -\sqrt{b_1 b_2}\sum_{l=1}^n u_{1l}^n \int_a^b e_{l,x}e_k dx - a_2\sum_{l=1}^n u_{2l}^n \int_a^b e_{l,x}e_{k,x}dx, \tag{6.13b}$$

where we have assumed that $(e_l,e_k)_{L_2(a,b)}=\delta_{lk}$. Since Eq. (6.13) is a system of ordinary linear equations with constant coefficients, we have the existence of solutions, so there exists an approximate solution, namely, solution (6.12), which satisfies energy inequality (6.9) and integral identity (6.10). Hence, after an appropriate passing with $n\to\infty$, we obtain the existence of weak solutions to Eq. (6.3) satisfying integral identity (6.10) and estimate (6.11). This concludes the proof. □

The proof of lemma 6.4 is sketched out. For more details, see Ladyzhenskaya *et al.* ([25], Chap. 3).

Finally, we increase regularity of weak solutions to get classical solutions of Eq. (6.3). The proof is also sketched (see Ref. [25]).

Lemma 6.5. Assume that $u_0\in H^3(a,b)$. Then there exists a solution to problem (6.3) such that $u\in W_2^{4,2}[(a,b)\times(0,T)]$ and the estimate

$$\|u\|_{W_2^{4,2}[(a,b)\times(0,T)]} \le c\|u_0\|_{H^3(a,b)} \tag{6.14}$$

holds.

Proof. From the existence of the weak solution and estimate (6.11) we have that $u_{,x}\in L_2[(a,b)\times(0,T)]$. Now we consider the parabolic problem

$$u_{1,t} - a_1 u_{1,xx} = -\sqrt{b_1 b_2}u_{2,x} \text{ in } (a,b)\times(0,T), \tag{6.15a}$$

$$u_{2,t} - a_2 u_{2,xx} = -\sqrt{b_1 b_2}u_{1,x} \text{ in } (a,b)\times(0,T), \tag{6.15b}$$

$$u_i|_{x\in\{a,b\}} = 0, \quad i=1,2, \text{ on } (0,T),$$

$$u_i|_{t=0} = u_{i0}, \quad i=1,2, \text{ in } (a,b).$$

Since $u_{,x}\in L_2[(a,b)\times(0,T)]$, we assume additionally that $u_0\in H^1(a,b)$. Then there exists a solution to problem (6.15) such that $u\in W_2^{2,1}[(a,b)\times(0,T)]$ and the estimate

$$\|u\|_{W_2^{2,1}[(a,b)\times(0,T)]} \le c\|u_0\|_{H^1(a,b)}$$

holds. By imbedding we have that $u_{,x}\in W_2^{1,1/2}[(a,b)\times(0,T)]$ and

$$\|u_{,x}\|_{W_2^{1,1/2}[(a,b)\times(0,T)]} \le c\|u\|_{W_2^{2,1}[(a,b)\times(0,T)]}.$$

Then the right-hand side of Eq. (6.15) belongs to $W_2^{1,1/2}[(a,b)\times(0,T)]$. Next, assuming that $u_0\in H^2(a,b)$, we have the existence of solutions to Eq. (6.15) such that $u\in W_2^{3,3/2}[(a,b)\times(0,T)]$ and the estimate

$$\|u\|_{W_2^{3,3/2}[(a,b)\times(0,T)]} \le c\|u_0\|_{H^2(a,b)} \tag{6.16}$$

is valid. From Eq. (6.16) it follows that $u_{,x}\in W_2^{2,1}[(a,b)\times(0,T)]$, so the right-hand side of Eq. (6.15) belongs to $W_2^{2,1}[(a,b)\times(0,T)]$. Hence for $u_0\in H^3(a,b)$ there exists a solution to problem (6.15) such that $u\in W_2^{4,2}[(a,b)\times(0,T)]$ and

$$\|u\|_{W_2^{4,2}[(a,b)\times(0,T)]} \le c\|u_0\|_{H^3(a,b)}.$$

This concludes the proof. □

Remark 6.6. From Eq. (6.14) we have that $u_{,t},u_{,xx}\in C^{\alpha,\alpha/2}[(a,b)\times(0,T)]$, where $\alpha<1/2$. Hence our initial-boundary value problem has a classical solution.

The conclusions are as follows. The system comprised of Eqs. (6.1a), (6.1b), and (6.2) is parabolic. This system was derived from Eq. (5.1) by setting $p=1$ and using Eqs. (5.12) and (5.13). Equation (4.7) with $p=1$ corresponds to the special case $d=0$. The cases $d=0$ and $d=1$ are interesting in that the mathematical structure of the model changes significantly (from parabolic type to a mixed hyperbolic-parabolic type) and existence of solutions is by no means obvious. Put somewhat differently, it is not clear how to formulate conditions on the problem data such that the existence and uniqueness of classical solutions is guaranteed. When $d\in(0,1)$, problem (6.1) has a unique classical solution with the following extra property:

$$\frac{n}{(n+1)^2}\int_a^b (z_1^2+z_2^2)dx \le e^{-2c_1 a_* t}\int_a^b (z_{10}^2+z_{20}^2)dx,$$

$$t \in (0,T),$$

where $T>0$ is arbitrary and

$$a_* = \begin{cases} \varepsilon(n+1)d/[3(n+2)] & \text{if } d \le 1/2, \\ \varepsilon(n+1)(1-d)/[3(n+2)] & \text{if } d > 1/2. \end{cases}$$

In the class of solutions subject to conditions (6.1c) and (6.1d), this implies that the zero solution of Eqs. (6.1a) and (6.1b) is asymptotically stable (in fact, exponentially stable). Note that $\int_a^b (z_1^2+z_2^2)dx$ is the Lyapunov functional for Eqs. (6.1a) and (6.1b). The method of Lyapunov, which played an important role in the stability theory of ordinary differential equations, has been extended to a number of problems described by partial differential equations [27]. Here this method was used to prove some results on the well posedness of parabolic systems arising in the variable transformation approach.

Equation (5.1) with $p=2$ is not parabolic. The Chapman-Enskog expansion of Eq. (3.5) does not lead to parabolic equations when $p=1,2$. In order to derive the parabolic system, it was necessary to assume that $p=1$ and subsequently use the method of transformations of variables.

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APPENDIX: SOME USEFUL SPACES

By $W_2^{k,l}[(a,b)\times(0,T)]$, $k,l \in \mathbb{N}_0=\mathbb{N}\cup\{0\}$, we denote a space of functions with the finite norm

$$\|u\|_{W_2^{k,l}[(a,b)\times(0,T)]} = \left[\sum_{\alpha\le k}\int_0^T dt\int_a^b dx|\partial_x^\alpha u(t,x)|^2 + \sum_{\beta\le l}\int_0^T dt\int_a^b dx|\partial_t^\beta u(t,x)|^2\right]^{1/2}.$$

We set $L_2[(a,b)\times(0,T)]=W_2^{0,0}[(a,b)\times(0,T)]$ and define $V_2^0[(a,b)\times(0,T)]$ as a space with the finite norm

$$\|u\|_{V_2^0[(a,b)\times(0,T)]} = \operatorname*{ess\,sup}_{t\in(0,T)}\|u(\cdot,t)\|_{L_2(a,b)} + \|\nabla u\|_{L_2[(a,b)\times(0,T)]},$$

where $L_2(a,b)$ is the space consisting of all functions that are square integrable on (a,b). By $(\cdot,\cdot)_{L_2(a,b)}$ we denote the scalar product in $L_2(a,b)$. We do not distinguish between scalar- and vector-valued functions and their corresponding norms. We denote by $W_2^k(a,b)$ the space having the norm

$$\|u\|_{W_2^k(a,b)} = \left(\sum_{\alpha\le k}\int_a^b dx|\partial_x^\alpha u|^2\right)^{1/2}.$$

The spaces $H^k(a,b)$ and $H_0^k(a,b)$ are defined as follows:

$$H^k(a,b) \equiv W_2^k(a,b), \quad H_0^k(a,b) \equiv \{u \in H^k(a,b); u|_{x\in\{a,b\}} = 0\}.$$

Next, we introduce spaces appropriate for parabolic equations. By $W_2^{l,l/2}[(a,b)\times(0,T)]$, l odd, we mean a Hilbert space with the norm

$$\|u\|_{W_2^{l,l/2}[(a,b)\times(0,T)]} = \left[\sum_{\alpha\le l}\int_0^T dt\int_a^b dx|\partial_x^\alpha u|^2 + \sum_{\beta\le[l/2]}\int_0^T dt\int_a^b dx|\partial_t^\beta u|^2 + \int_0^T dt'\int_0^T dt''\int_a^b dx\frac{|\partial_{t'}^{[l/2]}u(t',x)-\partial_{t''}^{[l/2]}u(t'',x)|^2}{|t'-t''|^2}\right]^{1/2}.$$

Here $[l/2]$ means the largest integer less than or equal to $l/2$. Finally, we denote by $C^{\alpha,\alpha/2}$, $\alpha\in(0,1)$, the space of Hölder continuous functions on $(a,b)\times(0,T)$. This space is equipped with the norm

$$\|u\|_{C^{\alpha,\alpha/2}} \equiv \max_{t,x}|u(t,x)| + \max_{t,x',x''}\frac{|u(t,x')-u(t,x'')|}{|x'-x''|^\alpha} + \max_{t',t'',x}\frac{|u(t',x)-u(t'',x)|}{|t'-t''|^{\alpha/2}}.$$

Note that $C^{\alpha,\alpha/2}$ is a Banach space.

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